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SMALL AMPLITUDE PRESSURE WAVE IN SUBSONIC AND SUPERSONIC BUBBLE FLOWS IN CONVERGENT-DIVERGENT NOZZLES

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Abstract—This paper makes a theoretical analysis of the propagation phenomena of the small amplitude pressure wave in the subsonic and supersonic bubble flow with a velocity slip between bubble and liquid in the convergent–divergent nozzle. From an analysis of the time-mean flow, the nondimensional parameter $m = \{u_G^2 \cdot \alpha(1-\alpha)\rho_L\beta(2-1/S)/P \cdot [\alpha\beta S + (1-\alpha)\beta S^2 + \alpha(1-\alpha)]\}^{1/2}$ corresponds to Mach number is gas-dynamics where u_G is the gas velocity, α: the void fraction, ρ_L: the liquid density, P: the pressure, S: the velocity ratio of the gas and liquid flows and β: the proportional constant for the virtual mass. From a theoretical analysis of the small disturbance field, it is clarified that the parameter m also plays an essential and important role as Mach number, although the propagation performance of the disturbance is very complicated compared with that in gasdynamics. It is also shown that the pressure waves are divided into four groups depending on the velocity ratio S. Two of them are rather realistic, but the other two are required of a further investigation in future.

1. INTRODUCTION

An analysis of the high speed bubble flow is important in connection with the flow-out accident of tube breaking of the water-cooled nuclear reactor and also with acceleration of a liquid by boiling of other volatile liquid in the liquid metal MHD power generator. A study of the propagation phenomena of pressure wave in the bubble flow where a relative speed exists between the gas and liquid phases shall be important in regard to behaviors of the pressure wave in the liquid lithium in the blanket of the pulse operating fusion reactor in the far future. Generally, the critical flow rate in a convergent–divergent nozzle and the propagating velocity of small amplitude pressure wave is closely related. For the inviscid gas flow, the relation between them is well known to be expressed by such the very simple form as $M = 1$ at the nozzle throat, where M is Mach number, to mean the gas critical velocity being equal to the sound one. On the other hand, in the two-phase bubble flow when a slip velocity exists between the phases, the definition of Mach number is not clear, nor has there been known any detailed investigation to give the relation between the critical flow condition and the propagating velocity of the pressure wave. It is considered that difficulties of the investigation lie in the fact that in the two-phase flow the large density of liquid and compressibility of gas are related with the pressure wave velocity, and that there exist two typical flow velocities; bubble velocity and liquid velocity. Many investigations have been done concerning the one-dimensional bubble flow, and typical studies are reviewed in Wijingaarden's report (1972). In his report, papers in regard to pressure waves in the bubble flow taking the dynamical behavior of bubble into consideration, flows in the Laval nozzle and waves of finite amplitude are summarised. But, the effect of the slip velocity between the bubble and liquid on the small amplitude pressure wave has not been considered. Although Wijingaarden considered the effect of the slip velocity on the nozzle flow in his review, the momentum equation for the fluid is considered not to be correct. Recently, Prosperetti & Van Wijingaarden (1976) analysed the characteristic velocity of the bubble flow in case of the slow relative velocity between the bubble and liquid flows, and compared them with the experimental value obtained from the critical flow rate reported by Muir & Eichhorn (1963).

In their calculation, the inertia force of liquid was used as the external force acting on the bubble, and the propagation phenomena of the pressure wave in the so called subsonic and supersonic bubble flow in a local point of the convergent–divergent nozzle were studied. The propagation velocity of small disturbance was theoretically analysed by using the small amplitude approximation, and the relation between the critical condition and the propagation velocity of disturbance was discussed. However it is considered that the assumption of small slip velocity is not always realized in the convergent–divergent nozzle, especially in the divergent section.

The purpose of this paper is to make a theoretical investigation of the propagation phenomena of the small amplitude pressure wave in general. It is clarified that a parameter corresponding to Mach number in gasdynamics is introduced as the most important and essential one and that quite different from the performance of pressure waves in gasdynamics the pressure waves in bubble flows are divided into four groups depending on the slip velocity between the bubble and liquid flows.

2. FUNDAMENTAL EQUATIONS

A one-dimensional bubble flow in a channel such as the convergent-divergent nozzle with the sectional area $A(x)$ is considered, the x axis being taken in the flow direction. Bubble radii r are uniform in a cross section and bubbles are considered to disperse homogeneously in the flow, and not to coalesce with each other. The conservation law for the bubble number density n is expressed by

$$\frac{D}{Dt_G}(nA) = -nA\frac{\partial u_G}{\partial x}, \qquad [1]$$

where t is time. The derivative of the left term expresses the substantial derivative with respect to the gas phase velocity u_G and is expressed as follows

$$\frac{D}{Dt_G} = \frac{\partial}{\partial t} + u_G\frac{\partial}{\partial x}. \qquad [2]$$

Assuming this bubble flow as a two-component flow, the mass conservation law of gas phase is expressed as

$$\frac{D}{Dt_G}\left(\frac{4}{3}\pi r^3 \rho_G\right) = 0, \qquad [3]$$

where ρ_G is the density of gas. By taking the sectional mean of this equation and using void fraction $\alpha(= n\cdot\frac{4}{3}\pi r^3)$, we have

$$\frac{\partial}{\partial t}(\alpha\rho_G A) + \frac{\partial}{\partial x}(\alpha\rho_G u_G A) = 0. \qquad [4]$$

Next, the equation of motion of the bubble is considered. When the Reynolds number (Re) based on the relative velocity is less than 10^3, the bubble keeps a spherical shape, and the following equation is obtained,

$$\frac{D}{Dt_G}\left(\frac{4}{3}\pi r^3 \rho_G u_G\right) = -\frac{4}{3}\pi r^3\frac{\partial P}{\partial x} - \frac{4}{3}\pi r^3\beta\rho_L\frac{D}{Dt_G}(u_G - u_L)$$
$$-4\pi r^2\beta\rho_L(u_G - u_L)\frac{Dr}{Dt_G} - \frac{C_D}{2}\pi r^2\rho_L(u_G - u_L)^2, \qquad [5]$$

where u_L is the liquid velocity, P: the pressure, ρ_L: the liquid density, and C_D: the resistence coefficient of bubble being expressed by 48/Re when Re is less than 10^3. The terms of [5] excepting the last one of the r.h.s. are obtained under the assumption of potential flow around a sphere which is reasonably accepted in the flow around a spherical gas bubble (Harper 1972). The full derivation of the equation is shown in the Appendix. The l.h.s. of [5] is the inertia force of gas phase which can be neglected compared with the other terms in the bubble flow region. The first term in the r.h.s. is the force acting on the bubble due to the pressure gradient and is replaced by the inertia force of liquid $\frac{4}{3}\pi r^3 \rho_L\ (Du_L/Dt_G)$ as shown in Prosperetti (1976). The second term is the force acting on the bubble by virtual mass caused by change of the relative motion, and the third term is that by another virtual mass effect caused by the radius increase rate where β is a coefficient, and is 1/2 for a single bubble in infinite liquid (Lamb 1932). The dependence of β on the bubble interval is reported by Van Wijingaarden (1976), but its dependency is weak, and β is assumed constant. In the two-phase flow in the convergent–divergent nozzle, because of the rapid change of sectional area of the nozzle, the forces by virtual mass are larger compared with the viscous force, therefore, the last term in the r.h.s. of [5] can be neglected.

Then, [5] is expressed as follows, using a void fraction α,

$$\frac{\partial}{\partial t}\{\alpha\beta\rho_L(u_G-u_L)A\}+\frac{\partial}{\partial x}\{\alpha\beta\rho_L(u_G-u_L)u_GA\}-\alpha A\frac{\partial P}{\partial x}=0. \qquad [6]$$

The continuity equation of liquid phase is given as

$$\frac{\partial}{\partial t}\{(1-\alpha)\rho_LA\}+\frac{\partial}{\partial x}\{(1-\alpha)\rho_Lu_LA\}=0. \qquad [7]$$

The momentum equation of the whole two-phase flow is expressed as,

$$\frac{\partial}{\partial t}(\alpha\rho_Gu_GA)+\frac{\partial}{\partial t}\{(1-\alpha)\rho_Lu_LA\}+\frac{\partial}{\partial x}(\alpha\rho_Gu_G^2A)+\frac{\partial}{\partial x}\{(1-\alpha)\rho_Lu_L^2A\}\frac{\partial P}{\partial x}=0. \qquad [8]$$

Being compared with the pressure gradient force, the wall friction can be neglected. Since the ratio of mass flow rate of gas phase to that of liquid phase is very small, the first and the 3rd terms of [8] can be neglected. As thermal conductivity and heat capacity of liquid are much larger than those of gas, the gas temperature considered to be kept equal to the liquid temperature, which is assumed constant. Therefore, the equation of state of gas is

$$\frac{P}{\rho_G}=\text{constant}. \qquad [9]$$

It has not been made clear whether this assumption of isothermal process is applicable for the whole frequency range. Plesset (1964) pointed out it can be accepted for very high and low frequency ranges. In this report, as the frequency dependence of acoustic velocity is not discussed, the assumption of isothermal process is adapted. For liquid phase, the density ρ_L is assumed constant.

3. STEADY NOZZLE FLOW

In an analysis of the relation between the flow velocity and the pressure wave propagating velocity, the high speed bubble flow is taken up and the velocity slip effect is taken into account, because it predominates in a highly accelerating flow such as the convergent–divergent nozzle flow. Assuming a steady flow and the shape of nozzle being given by a function of x, [4] and [6]–[8] construct the simultaneous differential equations for u_G, u_L, α and P, and they are given as

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln u_G = -\frac{1}{1-m^2}\left[\frac{\beta S+(1-\alpha)}{(1-\alpha)\beta S^2+\alpha\beta S+\alpha(1-\alpha)}-\frac{m^2(S-1)}{\alpha(2S-1)}\right]\frac{\mathrm{d}}{\mathrm{d}x}\ln A. \quad [10]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln u_G = -\frac{1}{1-m^2}\left\{\frac{\beta S^2}{(1-\alpha)\beta S^2+\alpha\beta S+\alpha(1-\alpha)}\right\}\frac{\mathrm{d}}{\mathrm{d}x}\ln A. \quad [11]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln \alpha = -\frac{1}{1-m^2}\left[\frac{(1-\alpha)\{S(S-1)\beta-(1-\alpha)\}}{(1-\alpha)\beta S^2+\alpha\beta S+\alpha(1-\alpha)}+\frac{m^2(1-\alpha)}{\alpha}\right]\frac{\mathrm{d}}{\mathrm{d}x}\ln A. \quad [12]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln P = \frac{1}{1-m^2}\left\{\frac{m^2 S}{\alpha(2S-1)}\right\}\frac{\mathrm{d}}{\mathrm{d}x}\ln A. \quad [13]$$

Here, the velocity ratio S is u_G/u_L, and the nondimensional parameter m is given as

$$m = \left[\frac{u_G^2\rho_L\alpha(1-\alpha)\beta(2S-1)}{PS\{(1-\alpha)\beta S^2+\alpha\beta S+\alpha(1-\alpha)\}}\right]^{1/2}. \quad [14]$$

On the other hand, in case of the gas flow in a nozzle, corresponding to [10] and [13] we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln u_G = -\frac{1}{1-M^2}\frac{\mathrm{d}}{\mathrm{d}x}\ln A. \quad [15]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln P = \frac{\gamma M^2}{1-M^2}\ln A. \quad [16]$$

Here, M is the Mach number, and γ is the ratio of specific heats.

Comparing [15] and [16] with [10]–[13], it is necessary for the term $1-m^2$ in the denominators of the fundamental equations to vanish at the throat of the nozzle. However, there leaves a question whether the nondimensional parameter m just given by [14] expresses the Mach number which is defined as the ratio of flow velocity and sound velocity or pressure wave propagating velocity. Also, it is not sure whether $m < 1$ in the convergent section upstream the throat means subsonic nor is it sure whether $m > 1$ in the divergent section means supersonic. This problem will be discussed afterwards. If the fundamental equations [10]–[13] are rewritten in terms of pressure changes instead of the axial coordinate, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln u_G = \frac{\alpha(2S-1)}{m^2 S}\left\{\frac{\beta S+(1-\alpha)}{(1-\alpha)\beta S^2+\alpha\beta S+\alpha(1-\alpha)}-\frac{m^2(S-1)}{\alpha(2S-1)}\right\}\frac{\mathrm{d}}{\mathrm{d}x}\ln P, \quad [17]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln A = \frac{(1-m^2)\alpha(2S-1)}{m^2 S}\frac{\mathrm{d}}{\mathrm{d}x}\ln P, \quad [18]$$

and singularity at $m = 1$ can be avoided. Instead, the condition to make $m = 1$ at the throat must be added.

By making use of [17] and [18] numerical calculation can be carried out for a convergent–divergent nozzle. In place of the ideal inlet condition such as $m_i = 0$, $A_i/A^* = \infty$ at the stagnation condition, a calculation was started at the point of $A_i/A^* = 100$, where A_i and A^* are the sectional areas of the nozzle at the inlet and the throat, respectively. At this point m^2 is approx. 10^{-4}. By taking the velocity ratio S as unity and by making the iterative assumption of the nondimensional parameter m at this inlet, calculation was made along the nozzle until $m = 1$ was satisfied at the throat. This result for the inlet void fraction $\alpha_i = 0.1$ is shown in figure 1 as a typical example. The horizontal axis is the nondimensional value m and in the vertical axis u_L/u_L^*, u_G/u_L^*, P/P^* and A/A^* are taken when the asterisk shows the value at the throat. The gas velocity was made nondimensional by the liquid velocity at the throat to prove the existence of velocity difference in the nozzle flow. Decrease in u_G/u_L^* at the divergent part, as seen in figure 1, is due to increase in the force of virtual mass due to the rapid bubble expansion by pressure decrease. These bubble and liquid velocity distributions well correspond to those of Muir's experimental results (1963).

4. PROPAGATION OF PRESSURE WAVE

Velocity of small amplitude wave propagating in a steady flow is analysed by the assumption of a one dimensional field. Hydrodynamical values are divided into mean and perturbed terms and are written as follows:

$$u_G(x, t) = u_{G_0}(x) + u_G' \exp\{i(kx - \omega t)\}, \quad [19]$$

$$u_L(x, t) = u_{L_0}(x) + u_L' \exp\{i(kx - \omega t)\}, \quad [20]$$

$$\alpha(x, t) = \alpha_o(x) + \alpha' \exp\{i(kx - \omega t)\}, \quad [21]$$

$$P(x, t) = P_o(x) + P' \exp\{i(kx - \omega t)\}, \quad [22]$$

where ω is the angular velocity of disturbance and k is the complex wave number. Substituting the above equation [4], [6], [7] and [8], the equations for mean values and perturbed values are obtained respectively. The equations for mean values are identical with fundamental equations mentioned in the preceding chapter.

When the wave length becomes of the same order as the length of the nozzle, the assumption of the one dimensional flat wave of small disturbance is broken. As the characteristic length of the nozzle is considered to be $[\mathrm{d} \ln A/\mathrm{d}x]^{-1}$, the following condition must be introduced to the wave number.

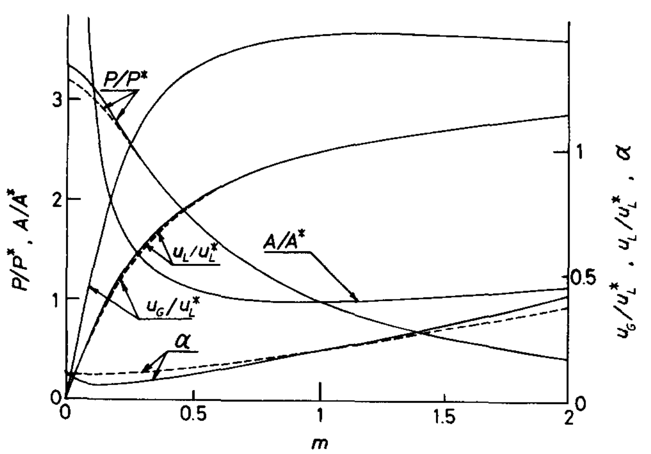

Figure 1. Relations among nondimensional pressure (P/P^*), nondimensional gas velocity (u_G/u_L^*), nondimensional liquid velocity (u_L/u_L^*), nondimensional sectional area (A/A^*), void fraction (α) and nondimensional parameter m along the convergent–divergent nozzle when $\alpha_i = 0.1$ and $\beta = 0.5$ (----) and $\beta \to \infty$ (——). $m < 1$, $m = 1$ and $m > 1$ are corresponding to convergent part, throat and divergent part of the nozzle, respectively.

$$|k| \gg \left[\frac{\mathrm{d}}{\mathrm{d}x}\ln A\right]. \qquad [23]$$

Since the wave length is short under this condition, the derivative of the mean values to the flow direction can be neglected and in the equations for the perturbed valued and the complex wave number has only the real part, because the imaginary part expresses the attenuation in the flow direction.

Under such conditions as mentioned above, [4], [6], [7] and [8] for the perturbated terms are linearized by use of [19], [20] and [22] as follows:

$$(-\omega + ku_{G_0})\frac{\alpha'}{\alpha_0} + (-\omega + ku_{G_0})\frac{P'}{P_0} + ku_{L_0}\frac{u'_G}{u_{L_0}} = 0 \qquad [24]$$

$$\left(\frac{u_{G_0}}{u_{L_0}} - 1\right)(-\omega + ku_{G_0})\frac{\alpha'}{\alpha_0} + \{(2u_{G_0} - u_{L_0})k - \omega\}\frac{u'_G}{u_{L_0}} - \{-\omega + ku_{G_0}\}\frac{u'_L}{u_{L_0}} + \frac{kP'}{\beta u_{L_0}\rho_L} = 0, \qquad [25]$$

$$-\left(\frac{\alpha_0}{1-\alpha_0}\right)(-\omega + ku_{L_0})\frac{\alpha'}{\alpha_0} + ku_{L_0}\frac{u'_L}{u_{L_0}} = 0, \qquad [26]$$

$$(-\omega + ku)_{L_0}\frac{u'_L}{u_{L_0}} + \frac{kP'}{(1-\alpha_0)\rho_L u_{L_0}} = 0. \qquad [27]$$

In [24]–[27], such the condition that α', P', u'_G and u'_L have non-trivial solutions is given by the following equations.

$$(-\lambda+1)(-S\lambda+1)^2\left(-\lambda+2-\frac{1}{S}\right) - \frac{(2S-1)}{m^2\{\alpha_0\beta S^2 + (1-\alpha_0)\beta S^3 + \alpha_0(1-\alpha_0)S\}}$$

$$\cdot\{\alpha_0\beta S(-S\lambda+1)(-\lambda+1) + (1-\alpha_0)\beta S^2(-\lambda+1)^2 + \alpha_0(1-\alpha_0)(-S\lambda+1)^2\} = 0, \qquad [28]$$

where

$$\lambda = \omega/(ku_{G_0}) = C/u_{G_0}, \qquad [29]$$

which expresses the propagation velocity C of small amplitude wave made nondimensional by the gas velocity. S and m in [28] are those used in the preceding section and express the velocity ratio and the nondimensional parameter defined by [14], respectively. The propagation velocity C can be obtained from [28] if the void fraction α_0, velocity ratio S and nondimensional parameter m are given at a point of the flow. The propagation velocities of such a small amplitude wave obtained above are identical with that obtained as the characteristics $\mathrm{d}x/\mathrm{d}t$ of the unsteady partical differential equations [4] and [6]–[8]. To clarify the features of the propagation velocity of small amplitude wave, first, a simple case is taken up. When $S = 1$, [28] becomes very simple and the solution is given as follows:

$$\lambda = 1, \quad \text{(equal root)}, \quad \lambda = 1 \pm \frac{1}{m_{S=1}}, \qquad [30]$$

whereas

$$m_{s=1} = m_{eq} = \left[\frac{\rho_L u_{G_0}{}^2\alpha_0(1-\alpha_0)}{P_0\{1-(\alpha_0(1-\alpha_0)/\beta)\}}\right]^{1/2}. \qquad [31]$$

Putting $m_{eq} = u_G/a$, and by expressing [30] by the dimensional propagation velocity C, we have

$$C = u_{G_0} = u_{L_0}, \quad C = u_{G_0} \pm a, \qquad [32]$$

whereas

$$a = \left[\frac{P_0\{1 + (\alpha_0(1-\alpha_0)/\beta)\}}{\rho_l\alpha_0(1-\alpha_0)}\right]^{1/2}. \qquad [33]$$

The former duplicated solution in [32] is a trivial solution having a main flow velocity and the latter express the waves propagating in the downstream and upstream directions riding on the flow. The value of a by [33] agrees with the sound velocity in the flow field without a uniform flow as reported by Crespo (1969) and by Hijikata & Mori (1974). It is well understood that the propagation velocities expressed by the latter form of [32] means that the superposition of flow and sound velocities is realized since a small amplitude approximation is used in the analysis.

Under the same condition, Prosperetti (1976) gave the following propagation velocity of small amplitude wave

$$a = \left[\frac{P_0\{1 + (\alpha_0/\beta)\}}{\rho_L\alpha_0(1-\alpha_0)}\right]^{1/2}. \qquad [34]$$

The difference between [33] and [34] is caused by the different expressions of the external force on the bubbles as already mentioned.

Secondly, the case when $S \neq 1$ is going to be discussed. In the nozzle flow, as mentioned above, there is a velocity difference between gas and liquid phases in general. In this case, the exact solutions of [28] must be obtained. The l.h.s. of [28] is rewritten as the 4th power equation of λ as follows;

$$f(\lambda) = b_0 + b_1\lambda + b_2\lambda^2 + b_3\lambda^3 + b_4\lambda^4, \qquad [35]$$

where

$$b_0 = \left(2 - \frac{1}{S}\right)\left(1 - \frac{1}{m^2}\right),$$

$$b_1 = -4S - 1 + \frac{1}{S} + 2\left(2 - \frac{1}{S}\right)\{\alpha_0\beta S\frac{(S+1)}{2} + (1-\alpha_0)\beta S^2 + \alpha_o(1-\alpha_o)S\}/[m^2\{\alpha_o\beta S + (1-\alpha_o)\beta S^2 + \alpha_o(1-\alpha_o)\}],$$

$$b_2 = 2S^2 + 5S - 1 - (2 - \frac{1}{S})S^2\{\beta + \alpha_o(1-\alpha_o)\}/[m^2\{\alpha_o\beta S + (1-\alpha_o)\beta S^2 + \alpha_o(1-\alpha_o)\}],$$

$$b_3 = -3S^2 - S,$$

$$b_4 = S^2.$$

One of the important parameters include in [35] is m, which is defined by [14] in the analysis of the mean flow field in the nozzle. It should be emphasized that the same parameter. m plays the important role in the analysis of performances of mean flow and disturbance in the bubble flow just as in the gas flow. Effects of m on the solution of λ are investigated in the following. As in the case of $S = 1/2$, the forces of virtual mass caused by the relative velocity and the force by expansion cancels each other, the effect of interia force of gas phase becomes relatively large. Hence the assumption introduced previously and used in the analysis becomes invalid. However, the case of $S = 1/2$ seldom occurs in practice, and S is usually larger than unity in the accelerating flow such as the one in the convergent–divergent nozzle as shown in figure 1, and in the following investigation cases for $S > 1/2$ are discussed.

For $m > 1$ and $S > 1/2$, the coefficients b_i of λ satisfy the following conditions:

$$b_0, b_2, b_4 > 0: \quad b_1, b_3 < 0. \qquad [36]$$

Taking into consideration the fact that the value at $\lambda = 0$ of the nth derivative of f with respect to λ is $n!b_n$, it is known from [36] that the real root of $f(\lambda) = 0$ is all in the domain of $\lambda > 0$. According to the definition of λ, the positive value of λ indicates the wave moving on to the downstream. Therefore, the fact mentioned above means that waves in the region of $m > 1$ all propagate downstream, and none propagates upstream. This fact corresponds to the performance of sound wave in the supersonic flow of gas.

When $m = 1$, b_0 in [35] becomes definitely zero independent of α_0 and β, and $\lambda = 0$ is the solution indicating the existence of stationary disturbance standing in the space. In the case of $m < 1$, as $b_0 < 0$ and $b_4 > 0$ it is seen that there exists at least one solution for $\lambda < 0$, meaning that there definitely exists a wave propagating upstream which corresponds to a sound wave going upstream in the subsonic gas flow.

As explained above and understood from behaviors of disturbance propagating in the two-phase flow field, it is seen that the nondimensional parameter m defined in [14] corresponds to the Mach number in the gas flow. Also, as seen from [10] through [13], for a steady one dimensional flow in the convergent–divergent nozzle, m is equal to unity when $dA/dx = 0$.

From these results, it may be appropriate to consider the nondimensional parameter m as Mach number in the two-phase bubble flow having velocity difference between gas and liquid phases. However, although the Mach number in the gas flow indicates the ratio of flow velocity to sound velocity, the parameter m in the two-phase flow may not be defined simply by the ratio of flow velocity to sound velocity. To make this point clear, the solution of [34] is numerically obtained and discussed in the following.

As the dependence of b_0, b_1, b_2, b_3, b_4 given by [35] on the void fraction is examined to have no peculiar or singular behaviors, a calculation is made in the case of $\alpha_0 = 0.5$ for an example.

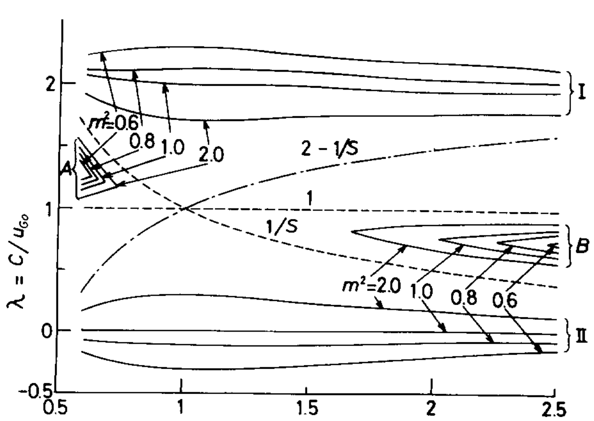

Figure 2. Relations among nondimensional propagation velocities (C/u_{G_0}), velocity ratio (u_{G_0}/u_{L_0}) and nondimensional parameter m are shown by solid lines (——) when $\alpha_o = 0.5$ and $\beta = 0.5$. Dotted lines (----) are corresponding to the equilibrium case ($\beta \to \infty$).

The result of $\lambda = C/u_G$ with m and the velocity ratio S as parameters is shown in figure 2 for $\beta = 1/2$. The vertical axis expresses the nondimensional propagation velocity λ, and the horizontal axis is the velocity ratio S of gas and liquid. The dotted lines in the figure are $\lambda = 1$ and $\lambda = 1/S$, which are the nondimensional gas and liquid velocities, respectively. There exist four kinds of waves as shown in figure 2, and two outside the lines of $\lambda = 1$ and $1/S$ and the other two between them. The formers are indicated by Groups I and II and the latters by Groups A and B.

Group I denotes waves moving faster than either gas velocity ($\lambda = 1$) or liquid velocity ($\lambda = 1/S$) and Group II slower ones. For $m > 1$, as can be seen from the figure, λ is always positive and the fast and slow waves go downstream. For $m < 1$, only the slow wave goes upstream as λ is negative. The wave of Group II going upstream for $m = 1$ is stationary relative to the nozzle as $\lambda = 0$.

There exist the other two wave groups indicated by Groups A and B in figure 2. Groups A and B expressed by the solid lines lying between $\lambda = 1$ and $1/S$ in the figure are of particular interest to us. The waves of these two groups move only downstream and do not appear unless the velocity difference of two phases becomes very large. In other words, in the case of small velocity difference, i.e. S being almost equal to unity, these waves do not appear unless m is very large. There have been no papers reporting on these Groups A and B. Further detailed theoretical or experimental studies shall be made in future.

With the increase of m, as seen in figure 2, the four groups of waves approach the curves of $\lambda = 1$, $1/S$ and $2 - 1/S$. Groups I and II and those of A and B correspond each other. By expressing the propagating velocity of the fast wave as C_+, that of the slow wave as C_-, the velocity of fluid carrying the waves as U and the propagating velocity of the waves relative to the fluid as V, we have

$$U = (C_+ + C_-)/2,$$
$$V = (C_+ - C_-)/2. \qquad [37]$$

As seen from figure 2, Groups I and II are carried down almost with the velocity of gas phase and Groups A and B with the intermediate velocity of gas and liquid phases. Although even in the high speed bubble flow in a conventional nozzle the slip ratio may not be much different from unity, these waves satisfying this condition are of interest and should be investigated in details in future.

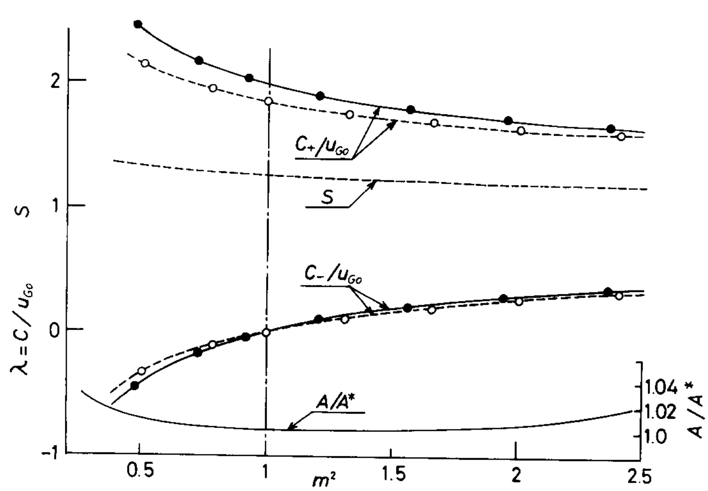

Figure 3. Relations among nondimensional propagation velocities (C/u_{G0}) and velocity ratio (u_{G0}/u_{L0}) and square of nondimensional parameter m along the nozzle when $\alpha_{oi} = 0.01$. Solid lines show the equilibrium case ($\beta \to \infty$). White circles (O) and dotted lines (----) and black circles (●) are obtained by numerical calculation for $\beta = 0.5$ and $\beta = 10^3$, respectively.

In figure 3, as an example, the propagation velocities of the fast and slow waves are calculated for the flow field of a convergent–divergent nozzle. The calculation was done using [17] and [18]. The horizontal axis indicates the nondimensional parameter m, while the vertical axis indicates the ratio on the cross section area (A/A^*) and the nondimensional propagation velocity λ. The solid line in the figure expresses the solution for $\beta = \infty$, i.e. $S = 1$, where the gas and liquid phases move with the same velocity. The black circle and the white circle indicate the numerical values for $\beta - 1000$ (not a practical value but taken up to study the effect of β) and 0.5 respectively. In the convergent part of the nozzle, m is smaller than unity and the slow wave propagates upstream $(\lambda < 0)$. In the divergent part $m > 1$ and the waves propagate only downstream $(\lambda > 0)$. Also, $m = 1$ holds at the throat of the nozzle and the solution of $\lambda > 0$ exists. In other words, for $m < 1$ the flow may be called subsonic, and for $m > 1$ supersonic. In this particular case, the velocity difference of the phases is not big enough, therefore the waves of Groups A and B do not appear.

5. CONCLUSION

In this paper a theoretical analysis of the subsonic and supersonic two-phase bubble flow in the nozzle with a velocity slip between phases has been made.

From an analysis of the time-mean flow it is found that the nondimensional parameter m defined by [14] including the void fraction and the velocity slip ratio S plays the same role as Mach number in gasdynamics. A numerical calculation for a convergent–divergent nozzle reveals that the force of virtual mass due to the rapid bubble expansion by pressure decrease is very important in the accelerating supersonic flow.

A theoretical analysis of the small disturbance field reduces a conclusion that m is also the most important parameter. The performance of the nondimensional propagation speed relative to the nozzle is divided into four groups depending on m and S as shown in figure 2. Groups I and II in the figure are realistic corresponding to the performance of the sound wave in gasdynamics. Groups A and B disappear when the velocities of two phases approach each other, i.e. S is nearly unity and have a peculiar characteristic which should be investigated in detail in future.

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APPENDIX

Forces on a sphere in a potential flow of a pressure gradient.

The following potential is proved to express an inviscid incompressible flow around an expanding sphere.

$$\phi = -u_o\frac{R^2}{r} - u_G\left[\frac{R^3}{2r^2}\cos\theta\right] - u_1\left[\left(\ell - \frac{r^2}{2\ell} + \frac{R^3}{\ell r} - \frac{2R^5}{3\ell r^3}\right) - \left(r + \frac{R^3}{2r^2}\right)\cos\theta + \left(\frac{3r^2}{2\ell} + \frac{R^5}{\ell r^3}\right)\cos^2\theta\right], \quad \text{[A1]}$$

where r and θ are spherical coordinates fixed at initial positions of the sphere, R is the radius of the sphere, u_o, u_G and u_L are the expanding velocity of the sphere, the uniform velocity of sphere and the main velocity at the initial position which is not constant along the flow direction to give the pressure gradient ℓ is the distance between the initial position of sphere and the source of the main flow. The velocity components in the r and θ directions are given as follows from [A1].

$$V_r = -u_L\left[\left(-\frac{r}{\ell} - \frac{R^3}{\ell r^2} + 2\frac{R^5}{\ell r^4}\right) - \left(1 - \frac{R^3}{r^3}\right)\cos\theta + \left(\frac{3r}{\ell} - \frac{3R^5}{\ell r^4}\right)\cos^2\theta\right] + u_G\left[\frac{R^3}{r^3}\cos\theta\right] + u_o\left[\frac{R^2}{r^2}\right],$$

$$V_\theta = -u_L\left[\left(1 + \frac{R^3}{2r^3}\right)\sin\theta - \left(3\frac{r}{\ell} + \frac{2R^5}{\ell r^4}\right)\cos\theta\sin\theta\right] + u_G\left[\frac{R^3}{2r^3}\sin\theta\right]. \quad \text{[A2]}$$

The velocities at the surface (V_{rR}, $V_{\theta R}$) and far from the sphere ($V_{r\infty}$, $V_{\theta\infty}$) are

$$V_{r\infty} = u_L\left(\frac{r}{\ell} + \cos\theta - 3\frac{r}{\ell}\cos^2\theta\right),$$

$$V_{\theta\infty} = u_L\left(-\sin\theta + 3\frac{r}{\ell}\cos\theta\sin\theta\right),$$

$$V_{rR} = u_G\cos\theta + u_o,$$

$$V_{r\theta} = u_L\left(-\frac{3}{2}\sin\theta + 5\frac{R}{\ell}\cos\theta\sin\theta\right) + \frac{u_G}{2}\sin\theta. \quad \text{[A3]}$$

It is easy to understand that the [A3] coincides with the velocities around the sphere located in the uniform inviscid flow when $\ell \to \infty$.

By considering the new coordinate whose origin is fixed at the center of the bubble, following relations are obtained geometrically,

$$\frac{\partial r}{\partial t} = u_G\cos\theta, \quad \frac{\partial\theta}{\partial t} = -\frac{u_G}{r}\sin\theta. \quad \text{[A4]}$$

Therefore, the partial derivative of the potential can be written as follows,

$$\frac{\partial\phi}{\partial t} = \frac{d\phi}{dt} - u_G\cos\theta\frac{\partial\phi}{\partial r} + \frac{u_G}{r}\sin\theta\frac{\partial\phi}{\partial\theta} \quad \text{[A5]}$$

where ϕ is given by [A1]. For the irrotational motion of a liquid, the pressure is expressed as follows,

$$\frac{P}{\rho}=\frac{\partial\phi}{\partial t}-\frac{1}{2}|V|^2+G(t). \qquad [A6]$$

The first and second terms in the r.h.s. of [A6] can be obtained from [A1], [A2] and [A5].

Here, the pressure is expanded into power series of $\cos\theta$ as

$$P = P_0(t)+P_1(t)\cos\theta+P_2(t)\cos^2\theta\ldots \qquad [A7]$$

If the pressure gradient in the flow direction is not so large, the terms higher than the first order may be neglected. As the zero order term has no effect on the force the flow direction acting on the sphere, we only consider the $P_1(t)$ of [A7]. From the calculation of the r.h.s. term of the [A6], P_1 at the surface of the sphere (P_{1R}) and far from the sphere ($P_{1\infty}$) are obtained as follows,

$$\frac{P_{1R}}{\rho}=-\frac{3R}{2}\frac{\mathrm{d}u_L}{\mathrm{d}t}+\frac{R}{2}\frac{\mathrm{d}u_G}{\mathrm{d}t}-u_o(u_L-u_G)\frac{3}{2},$$

$$\frac{P_{1\infty}}{\rho}=-R\frac{\mathrm{d}u_L}{\mathrm{d}t}, \qquad [A8]$$

where $u_o=\mathrm{d}R/\mathrm{d}t$. The force to the flow direction acting on the sphere is obtained from surface integration of [A8].

$$F_z=-\int_0^{\pi}P_{1R}\,2\pi R^2\sin\theta\cos\theta\,\mathrm{d}\theta$$

$$=\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{4}{3}\pi R^3(u_L-u_G)\right\}-\frac{4}{3}\pi R^3\left(\frac{\partial P}{\partial z}\right)_{r\to\infty}, \qquad [A9]$$

where z is the distance from the center of the sphere along the flow direction.